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Hierarchies of nonlinear integrable q-difference equations from series of Lax pairs

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Abstract
We present two hierarchies of nonlinear, integrable q-difference equations, one of which includes a q-difference form of each of the second and fifth Painlevé equations, $\mathrm{qP_{II}}$ and $\mathrm{qP_V}$, the other includes $\mathrm{qP_{III}}$. All the equations have multiple free parameters. A method to calculate a 2×2 Lax pair for each equation in the hierarchy is also given.

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1. Introduction

Nonlinear evolution equations occur frequently in physical modeling and applied mathematics. Nonlinear integrable lattice equations provide a natural discrete extension of classically integrable systems. For example, the lattice modified Korteweg–de Vries equation

$$\text{LMKdV:} \quad x_{l+1,m+1} = x_{l,m}\frac{(px_{l+1,m} - rx_{l,m+1})}{(px_{l,m+1} - rx_{l+1,m})}$$

provides an integrable discrete version of the well-known modified Korteweg–de Vries equation. More recently, there has been great interest in nonlinear ordinary difference equations because such q-discrete Painlevé equations are of fundamental interest in the theory of integrable systems and random matrix theory amongst other subjects. The integrability of such equations lies in the fact that they can be solved through an associated linear problem called a Lax pair.

Reductions constitute a natural connection between lattice equations and ordinary difference equations. The LMKdV equation is de-autonomized by allowing p and r to depend on l and m and there are known reductions from the non-autonomous LMKdV equation to q-discrete forms of the second, third and fifth Painlevé equations, denoted by $\mathrm{qP_{II}}$, $\mathrm{qP_{III}}$, and $\mathrm{qP_V}$, respectively [1, 2]. Different types of these reductions are possible, one of the simplest of which is to set $x_{l,m+1} = x_{n+d,m}$ for some positive integer d. In fact, the reductions that take the LMKdV equations to $\mathrm{qP_{II}}$ and $\mathrm{qP_V}$ are of this type with $d = 2$ and $d = 3$, respectively. There apparently exist an infinite series of such reductions that result in equations of arbitrary order, which suggests the existence of a hierarchy of equations. In a recent paper [2], we established a connection between these non-autonomous reductions and reductions of a Lax pair for the LMKdV equation itself. In this way, Lax pairs for non-autonomous versions of qP_{II}, qP_{III} and qP_{V} with multiple free parameters were discovered.

Here, we find Lax pairs for higher order equations and we thereby lay the groundwork for a hierarchy of equations. Two hierarchies are shown to exist: at the base of one lies qP_{II} and qP_{V}, while qP_{III} lies at the base of the other.

While there is a relatively large body of literature focusing on continuous Painlevé hierarchies [4–6] and some results concerning hierarchies of d-discrete nonlinear equations [3, 4], the problem of q-discrete hierarchies has been more elusive. Although a hierarchy of integrable nonlinear q-difference equations was found in [7], we believe that the results presented in this work are first example of such a hierarchy found by expansions of Lax pairs.

The paper is organized as follows. In section 2, we derive the formulae used to calculate quantities that exactly describe an equation in one of the hierarchies. These are given in terms of the same quantities describing the equation at a lower order and so we thereby obtain a recursive method of finding the hierarchies. We go further to purport a general formula that yields those quantities for any member of either hierarchy, and thus any equation in the hierarchies with its Lax pair. In section 3, we clarify the application of the formulae found in section 2 and use them to confirm a known result. In section 4, we derive new equations and Lax pairs. We end the paper with a conclusion where we also point out some open problems.

2. How to construct the hierarchy

In this section, we will establish the existence of two hierarchies of nonlinear, integrable, ordinary q-difference equations that are each obtainable from the LMKdV equation via a reduction. In section 2.1, the procedure for constructing the hierarchies will be derived in relation to the first hierarchy, which corresponds to reductions of the type $x_{l,m+1} = x_{l+d,m}$ for some integer d. Subsequently, in section 2.2 we shall outline an analogous process that leads to the second hierarchy corresponding to reductions of the type $x_{l,m+1} = 1/x_{l+d,m}$.

We establish the existence of the hierarchies by developing formulae that construct a member of the hierarchy from the preceding, lower order member. However, rather than iterating the equation or terms in the Lax pair directly, as has been the procedure used for some other systems [3], we will derive formulae for iterating a set of coefficients, introduced in equation (2.14), that describe the Lax pairs for the equations in the hierarchy.

2.1. Hierarchy corresponding to reductions of the type $x_{l,m+1} = x_{l+d,m}$

Begin with the linear problem

$$v(l+1,n) = L(l,n)v(l,n), \qquad v(l,n+1) = N(l,n)v(l,n), \tag{2.1}$$

whose compatibility condition is $L(l,n+1)N(l,n) = N(l+1,n)L(l,n)$. Hereafter, we adopt the notation $\bar{v} = v(l+1,n)$ and $\tilde{v} = v(l,n+1)$. Now set

$$L = \begin{pmatrix} \bar{x}/x & -k/(\lambda x) \\ -k\bar{x}/\lambda & 1 \end{pmatrix}, \tag{2.2a}$$

$$N = \begin{pmatrix} a_0 + a_2k^2 + \cdots + a_{2\rho}k^{2\rho} & b_1k + b_3k^3 + \cdots + b_{2\rho\pm1}k^{2\rho\pm1} \\ c_1k + c_3k^3 + \cdots + c_{2\rho\pm1}k^{2\rho\pm1} & d_0 + d_2k^2 + \cdots + d_{2\rho}k^{2\rho} \end{pmatrix}, \tag{2.2b}$$

where k is associated with the spectral variable n such that $k = k_0 q^n$, and x, λ and all of a_i, b_i, c_i, d_i are functions of l alone. The diagonal entries of the N matrix in the Lax pair contain only even powers of k, including a term constant in k, up to $k^{2\rho}$ where ρ is a positive integer. The off-diagonals of N contain only odd powers of k up to $k^{2\rho\pm1}$ depending on which part of the hierarchy we are considering.

Compatibility occurs when $\tilde{L}N = \overline{N}L$. It is not difficult to show that the compatibility condition can be written as follows

$$\bar{a}_i = a_i + \frac{1}{\lambda}\left(x\bar{b}_{i-1} - q\frac{c_{i-1}}{\bar{x}}\right), \qquad i \text{ even} \tag{2.3a}$$

$$x\bar{b}_i = \bar{x}b_i + \frac{1}{\lambda}(\bar{a}_{i-1} - qd_{i-1}), \qquad i \text{ odd} \tag{2.3b}$$

$$\frac{\bar{c}_i}{x} = \frac{c_i}{\bar{x}} + \frac{1}{\lambda}(\bar{d}_{i-1} - qa_{i-1}), \qquad i \text{ odd} \tag{2.3c}$$

$$\bar{d}_i = d_i + \frac{1}{\lambda}\left(\frac{\bar{c}_{i-1}}{x} - q\bar{x}b_{i-1}\right), \qquad i \text{ even} \tag{2.3d}$$

corresponding to entries (1, 1), (1, 2), (2, 1) and (2, 2), respectively. Some equivalences may be found between equations (2.3) if, at this point, we introduce the quantities

$$A_i = \begin{cases} a_i, & i \text{ even} \\ \bar{x}b_i, & i \text{ odd} \end{cases} \tag{2.4}$$

$$D_i = \begin{cases} d_i, & i \text{ even} \\ c_i/\bar{x}, & i \text{ odd,} \end{cases} \tag{2.5}$$

so that equations (2.3) become

$$\frac{x}{\bar{\bar{x}}}\overline{A}_{i-1} = \lambda(\overline{A}_i - A_i) + qD_{i-1}\,, \qquad i \text{ even} \tag{2.6a}$$

$$\overline{A}_{i-1} = \lambda\left(\frac{x}{\bar{\bar{x}}}\overline{A}_i - A_i\right) + qD_{i-1}\,, \qquad i \text{ odd} \tag{2.6b}$$

$$\overline{D}_{i-1} = \lambda\left(\frac{\bar{\bar{x}}}{x}\overline{D}_i - D_i\right) + qA_{i-1}\,, \qquad i \text{ odd} \tag{2.6c}$$

$$\frac{\bar{\bar{x}}}{x}\overline{D}_{i-1} = \lambda(\overline{D}_i - D_i) + qA_{i-1}\,, \qquad i \text{ even.} \tag{2.6d}$$

In equations (2.6) we may substitute

$$X_i = \left(\frac{\bar{\bar{x}}}{x}\right)^{\frac{1-(-1)^i}{2}} = \begin{cases} \bar{\bar{x}}/x, & i \text{ odd} \\ 1, & i \text{ even,} \end{cases}$$

so that either equation (2.6a) or (2.6b), with i even or odd, respectively, will become

$$\frac{\overline{A}_{i-1}}{X_{i-1}} = qD_{i-1} + \lambda\left(\frac{\overline{A}_i}{X_i} - A_i\right), \qquad \forall\, i \tag{2.7}$$

and similarly equations (2.6c) and (2.6d), with i odd or even, respectively, become

$$\overline{D}_{i-1}X_{i-1} = qA_{i-1} + \lambda(\overline{D}_iX_i - D_i), \qquad \forall\, i. \tag{2.8}$$

By repeated use of equations (2.7) and (2.8), respectively, we arrive at the following:

$$\overline{A}_i = X_i\left[qD_i - \sum_{j=i+1}^{m} \lambda^{j-i}(A_j - qD_j)\right] \tag{2.9a}$$

$$\overline{D}_i = \frac{1}{X_i}\left[qA_i - \sum_{j=i+1}^{m} \lambda^{j-i}(D_j - qA_j)\right], \tag{2.9b}$$

where m is equal to the greatest degree of the polynomials in k located in the entries of the N matrix (2.2b), i.e. m is either 2ρ or $2\rho + 1$. Now add $q/X_i \times$ (2.9a) to $X_i \times$ (2.9b) so that

$$\frac{q}{X_i}\bar{A}_i + X_i\bar{D}_i = q^2 D_i + qA_i + \sum_{j=i+1}^{m} (q^2 - 1)\lambda^{j-i} D_j. \tag{2.10}$$

However, at $i = 0$, (2.6) shows that both A_0 and D_0 are constant, meaning that (2.10) at $i = 0$ becomes

$$-D_0 = \sum_{j=1}^{m} \lambda^j D_j,$$

which we can rearrange so that

$$-D_1 = D_0/\lambda + \sum_{j=2}^{m} \lambda^{j-1} D_j. \tag{2.11}$$

Similarly, q(2.9a) + (2.9b) gives us an expression for A_1:

$$-A_1 = A_0/\lambda + \sum_{j=2}^{m} \lambda^{j-1} A_j. \tag{2.12}$$

These expressions for A_1 and D_1, (2.11) and (2.12), can be substituted back into (2.7) and (2.8) to find expressions for A_2 and D_2 in terms of A_i, D_i with $i > 2$. We can continue this process to successively calculate all the terms in the Lax pair, A_i and D_i, thus resolving the Lax pair for a particular value of m. However, in the interest of establishing the existence of a hierarchy of equations, we will proceed to derive a formula for calculating successive iterates from previous ones. To do this we first rewrite (2.9a) as

$$-\bar{A}_i + qX_iD_i + X_i \sum_{j=i+1}^{m} \lambda^{j-i}(qD_j - A_j) = 0. \tag{2.13}$$

We aim to calculate each of the quantities in the Lax pair, A_i, D_i, in terms of the remaining quantities, $A_j, D_j \forall\ j > i$ and A_0, D_0. In general any A_i of interest might be found in terms of all A_j and D_j, $\forall\ j > i$. However, when the calculations are performed, it is observed that terms A_i only depend an terms A_j (not D_j) so it is conjectured that we can write any A_i or D_i as

$$-A_i = \alpha_0^i A_0 + \sum_{j=i+1}^{m} \alpha_j^i A_j \tag{2.14a}$$

$$-D_i = \delta_0^i D_0 + \sum_{j=i+1}^{m} \delta_j^i D_j, \tag{2.14b}$$

where we have introduced a series of coefficients α^i_j and δ^i_j that need to be found. Substitute the expansion (2.14a) into (2.9a), noting that $A_0 =$ constant, to get

$$\bar{\alpha}^i_0 A_0 + \sum_{j=i+1}^{m} \bar{\alpha}^i_j \bar{A}_j + qX_i D_i + X_i \sum_{j=i+1}^{m} \lambda^{j-i}(qD_j - A_j) = 0,$$

which, upon exploiting (2.9a) again to replace $\overline{A}_j$, becomes

$$\begin{aligned}
&\bar{\alpha}^i_0 A_0 - qX_i\delta^i_0 D_0 + \sum_{j=i+1}^{m} \bar{\alpha}^i_j X_j \sum_{k=j+1}^{m} \lambda^{k-j}(qD_k - A_k) \\
&\qquad + \sum_{j=i+1}^{m} \left\{ D_j q\left[X_i\left(\lambda^{j-i} - \delta^i_j\right) + \bar{\alpha}^i_j X_j\right] - \lambda^{j-i} A_j X_i \right\} = 0.
\end{aligned}$$

Now we may rearrange the double sum to arrive at the following:

$$\begin{aligned}
&\bar{\alpha}^i_0 A_0 - qX_i\delta^i_0 D_0 + D_{i+1}q\left[X_i\left(\lambda - \delta^i_{i+1}\right) + \bar{\alpha}^i_{i+1}X_{i+1}\right] - \lambda A_{i+1}X_i \\
&\qquad + \sum_{j=i+2}^{m} \left\{ D_j q \left[X_i\left(\lambda^{j-i} - \delta^i_j\right) + \bar{\alpha}^i_j X_j + \sum_{k=i+1}^{j-1} \lambda^{j-k} X_k \bar{\alpha}^i_k \right] \right. \\
&\qquad \left. - A_j \left(\lambda^{j-i} X_i + \sum_{k=i+1}^{j-1} \lambda^{j-k} X_k \bar{\alpha}^i_k \right) \right\} = 0. \qquad (2.15)
\end{aligned}$$

A repeat of this process beginning with (2.9b) brings us to

$$\begin{aligned}
&\bar{\delta}^i_0 D_0 - q\frac{\alpha^i_0}{X_i} A_0 + A_{i+1} q \left[\frac{1}{X_i}(\lambda - \alpha^i_{i+1}) + \frac{\bar{\delta}^i_{i+1}}{X_{i+1}} \right] - \frac{\lambda D_{i+1}}{X_i} \\
&\qquad + \sum_{j=i+2}^{m} \left\{ A_j q \left[\frac{1}{X_i}(\lambda^{j-i} - \alpha^i_j) + \frac{\bar{\delta}^i_j}{X_j} + \sum_{k=i+1}^{j-1} \frac{\lambda^{j-k}\bar{\delta}^i_k}{X_k} \right] \right. \\
&\qquad \left. - D_j \left(\frac{\lambda^{j-i}}{X_i} + \sum_{k=i+1}^{j-1} \frac{\lambda^{j-k}\bar{\delta}^i_k}{X_k} \right) \right\} = 0. \qquad (2.16)
\end{aligned}$$

To calculate the next sets of coefficients α^{i+1}_j and δ^{i+1}_j, $j = i+2, \ldots, m$, we must combine equations (2.15) and (2.16) in the correct way. We claim that we can eliminate the quantities D_i by adding $q(2.16) + \bar{\delta}^i_0 / \left(\delta^i_0 X_i\right)$ (2.15), which tallies to

$$\begin{aligned}
&A_0 \left(\frac{\bar{\alpha}^i_0 \bar{\delta}^i_0}{X_i \delta^i_0} - q^2 \frac{\alpha^i_0}{X_i} \right) + A_{i+1} \left\{ q^2 \left[\frac{1}{X_i}\left(\lambda - \alpha^i_{i+1}\right) + \frac{\bar{\delta}^i_{i+1}}{X_i} \right] - \frac{\lambda \bar{\delta}^i_{i+1}}{\delta^i_0} \right\} \\
&\qquad + \sum_{j=i+2}^{m} A_j \left\{ q^2 \left[\frac{1}{X_i}\left(\lambda^{j-i} - \alpha^i_j\right) + \frac{\bar{\delta}^i_j}{X_j} + \sum_{k=i+1}^{j-1} \frac{\lambda^{j-k}\bar{\delta}^j_k}{X_k} \right] \right. \\
&\qquad \left. - \left(\frac{\lambda^{j-i}\bar{\delta}^i_0}{\delta^i_0} + \sum_{k=i+1}^{j-1} \frac{\lambda^{j-k} X_k \bar{\delta}^i_0 \bar{\alpha}^j_k}{X_i \delta^i_0} \right) \right\} = 0. \qquad (2.17)
\end{aligned}$$

It is not obvious that every D_i should be canceled out in the above sum but this occurs in every calculation performed to date and we conjecture that it is always the case. From here we can make $-A_{i+1}$ the subject and so find the sought after coefficients

$$-A_{i+1} = A_0 \frac{\bar{\alpha}_0^i \bar{\delta}_0^i - q^2 \alpha_0^i \delta_0^i}{q^2\left[\delta_0^i\left(\lambda - \alpha_0^i\right) + \frac{X_i}{X_{i+1}}\delta_0^i \bar{\delta}_{i+1}^i\right] - \lambda X_i \bar{\delta}_0^i}$$
$$+ \sum_{j=i+2}^{m} A_j \left\{ \frac{q^2\left[\delta_0^i\left(\lambda^{j-i} - \alpha_j^i\right) + \frac{\delta_0^i \bar{\delta}_j^i}{X_j} + \sum_{k=i+1}^{j-1} \frac{\lambda^{j-k}\delta_0^i \bar{\delta}_k^j X_i}{X_k}\right] - \left(\lambda^{j-i}\bar{\delta}_0^i X_i + \sum_{k=i+1}^{j-1} \lambda^{j-k}\bar{\delta}_0^i \bar{\alpha}_k^j\right)}{q^2\left[\delta_0^i\left(\lambda - \alpha_0^i\right) + \frac{X_i}{X_{i+1}}\delta_0^i \bar{\delta}_{i+1}^i\right] - \lambda X_i \bar{\delta}_0^i} \right\}. \tag{2.18}$$

Comparing (2.18) with (2.14a) shows that

$$\alpha_0^{i+1} = \frac{\bar{\alpha}_0^i \bar{\delta}_0^i - q^2 \alpha_0^i \delta_0^i}{q^2 \delta_0^i\left[\lambda - \alpha_{i+1}^i + \frac{X_i}{X_{i+1}}\bar{\delta}_{i+1}^i\right] - \lambda X_i \bar{\delta}_0^i} \tag{2.19a}$$

$$\alpha_j^{i+1} = \frac{1}{G_i(\boldsymbol{\alpha}, \boldsymbol{\delta}, \mathbf{X})} \left\{ q^2 \delta_0^i \left[\lambda^{j-i} - \alpha_j^i + \frac{\bar{\delta}_j^i X_i}{X_j} + \sum_{k=i+1}^{j-1} \frac{\lambda^{j-k}\bar{\delta}_k^i X_i}{X_k}\right] - \bar{\delta}_0^i \left(\lambda^{j-i} X_i + \sum_{k=i+1}^{j-1} \lambda^{j-k} \bar{\alpha}_k^i X_k\right) \right\}, \tag{2.19b}$$

where G_i in (2.19) is the same as the denominator in (2.19a). In fact, we shall say

$$\alpha_0^{i+1} = \frac{H_0^i(\boldsymbol{\alpha}, \boldsymbol{\delta}, \mathbf{X})}{G_i(\boldsymbol{\alpha}, \boldsymbol{\delta}, \mathbf{X})} \tag{2.20a}$$

$$\alpha_j^{i+1} = \frac{H_j^i(\boldsymbol{\alpha}, \boldsymbol{\delta}, \mathbf{X})}{G_i(\boldsymbol{\alpha}, \boldsymbol{\delta}, \mathbf{X})}. \tag{2.20b}$$

The H and G quantities in (2.20) are defined by comparison with (2.19) where we have introduced the bold face notation $\boldsymbol{\alpha}$ to signify all α, $\bar{\alpha}$, etc, with any superscripts and subscripts.

Naturally, we must also repeat the operations to find an expression for the other coefficients δ_j^i, this begins with adding q(2.15)+$\bar{\alpha}_0^i X_i / \alpha_0^i$ (2.16) and leads to

$$\delta_0^{i+1} = \frac{\bar{\delta}_0^i \bar{\alpha}_0^i - q^2 \delta_0^i \alpha_0^i}{q^2 \alpha_0^i\left[\lambda - \delta_{i+1}^i + \frac{X_{i+1}}{X_i}\bar{\alpha}_{i+1}^i\right] - \frac{\lambda \bar{\alpha}_0^i}{X_i}} \tag{2.21a}$$

$$\delta_j^{i+1} = \frac{1}{G_i(\boldsymbol{\delta}, \boldsymbol{\alpha}, 1/\mathbf{X})} \left\{ q^2 \alpha_0^i \left[\lambda^{j-i} - \delta_j^i + \frac{\bar{\alpha}_j^i X_j}{X_i} + \sum_{k=i+1}^{j-1} \frac{\lambda^{j-k}\bar{\alpha}_k^i X_k}{X_i}\right] - \bar{\alpha}_0^i \left(\lambda^{j-i}/X_i + \sum_{k=i+1}^{j-1} \frac{\lambda^{j-k}\bar{\delta}_k^i}{X_k}\right) \right\}. \tag{2.21b}$$

Note that the quantities H and G from (2.20) arise again in equations (2.21) but this time as

$$\delta_0^{i+1} = \frac{H_0^i\left(\boldsymbol{\delta}, \boldsymbol{\alpha}, \frac{1}{\mathbf{X}}\right)}{G_i\left(\boldsymbol{\delta}, \boldsymbol{\alpha}, \frac{1}{\mathbf{X}}\right)} \tag{2.22a}$$

$$\delta_j^{i+1} = \frac{H_j^i\left(\boldsymbol{\delta}, \boldsymbol{\alpha}, \frac{1}{\mathbf{X}}\right)}{G_i\left(\boldsymbol{\delta}, \boldsymbol{\alpha}, \frac{1}{\mathbf{X}}\right)}. \tag{2.22b}$$

Importantly, since $\alpha_j^1 = \delta_j^1$, (2.20) and (2.22) indicate that $\alpha_j^i(\boldsymbol{\lambda}, \mathbf{x}) = \delta_j^i\left(\boldsymbol{\lambda}, \frac{1}{\mathbf{x}}\right)$ for $i > 1$. Hence, we only need to calculate the coefficients α_j^i in practice as δ_j^i follow from these results.

Because these coefficients exactly describe a Lax pair and an associated nonlinear equation, we have shown that this system does indeed constitute a hierarchy by constructing a general operation that takes the members at any level of the hierarchy to the next level. To find a particular Lax pair in the hierarchy, we truncate the series at some point A_m, D_m say, and use the coefficients to calculate each of the terms A_i, D_i that appear in the N matrix of the Lax pair, the L matrix is always the same. The equation associated with any Lax pair can be found via the compatibility condition. We may also continue to find higher order members of the hierarchy by subsequently reinstating some of the terms A_ι, D_ι with $\iota > m$ and calculating the coefficients needed to describe those terms, α^i_j and δ^i_j, through equations (2.19) and (2.21).

2.2. *Hierarchy corresponding to reductions of the type* $x_{m+1,l} = 1/x_{m,l+d}$

The formulae for the coefficients that were found in the preceding section corresponded to equations that can be obtained from the LMKdV equation via the reduction $\hat{x} = x_{l+d}$. However, in [2] it was shown that reductions of the type $\hat{x} = 1/x_{l+d}$ can also be used and that these reductions lead to q-discrete Painlevé equations as well. The hierarchy of equations that springs from this type of reduction has Lax pairs that are very similar to those used in section 2.1 and fit easily into the present framework. The main difference between the two sets of Lax pairs can be described in terms of the spectral parameter k. The Lax pairs in section 2.1 all have the following form:

$$N = \begin{pmatrix} a_0 + a_2k^2 + \cdots + a_{2\rho}k^{2\rho} & b_1k + b_3k^3 + \cdots + b_{2\rho\pm1}k^{2\rho\pm1} \\ c_1k + c_3k^3 + \cdots + c_{2\rho\pm1}k^{2\rho\pm1} & d_0 + d_2k^2 + \cdots + d_{2\rho}k^{2\rho} \end{pmatrix}.$$

Observe that the terms that contain the lowest power of k, that is terms that are constant in k, appear in the diagonal entries of N. Also note that the other half of the Lax pair, the L matrix from equation (2.2a), remains the same for both types of reduction.

Moving now to the hierarchy associated with reductions of the type $\hat{x} = 1/x_{l+d}$, we find that the associated Lax pairs have a form similar to the former case, except here the lowest powers of k appear in the off-diagonal entries. This can be achieved simply by removing the constant terms from the diagonal entries:

$$N = \begin{pmatrix} a_2k^2 + \cdots + a_{2\rho}k^{2\rho} & b_1k + b_3k^3 + \cdots + b_{2\rho\pm1}k^{2\rho\pm1} \\ c_1k + c_3k^3 + \cdots + c_{2\rho\pm1}k^{2\rho\pm1} & d_2k^2 + \cdots + d_{2\rho}k^{2\rho} \end{pmatrix}.$$

We can find the hierarchy that arises from this case in a congruent manner to the last with only minor alterations. The differences here arise because A_1 is now the first term in the series but it is not a constant, as was A_0 in the previous case. Instead $A_1 = \beta_1 x\bar{x}$, $\beta_1 =$ constant, which introduces additional factors of $X_1 = \bar{\bar{x}}/x$ into the equations after (2.14a). Following the same procedure as in section 2.1, it is not difficult to show that in this case the formulae analogous to (2.19) are

$$\alpha^{i+1}_1 = \frac{\bar{\alpha}^i_1\bar{\delta}^i_1X_i^2 - q^2\alpha^i_1\delta^i_1}{q^2\delta^i_1\left[\lambda - \alpha^i_{i+1} + \frac{X_i}{X_{i+1}}\bar{\delta}^i_{i+1}\right] - \lambda X_1X_i\bar{\delta}^i_1} \tag{2.23a}$$

$$\begin{aligned}\alpha^{i+1}_j = \frac{1}{G_i}\Bigg\{ & q^2\delta^i_1\left[\lambda^{j-i} - \alpha^i_j + \frac{\bar{\delta}^i_jX_i}{X_j} + \sum_{k=i+1}^{j-1}\frac{\lambda^{j-k}\bar{\delta}^i_kX_i}{X_k}\right] \\ & - \bar{\delta}^i_1\left(\lambda^{j-i}X_1X_i + \sum_{k=i+1}^{j-1}\lambda^{j-k}\bar{\alpha}^i_kX_1X_k\right)\Bigg\},\end{aligned} \tag{2.23b}$$

where G_i in (2.23b) is the same as the denominator in (2.23a). We will present the first few equations in this hierarchy in section 4.2

2.3. General coefficients

Upon calculating a number of terms, a pattern appeared and it is thus conjectured that all the coefficients for the hierarchy of equations that arise from reductions of the type $x_{m+1,l} = x_{m,l+d}$ are given by the following equations:

$$\alpha_j^k = \sum_{i_1=0}^{j-k} \sum_{i_2=0}^{j-k-I_1} \cdots \sum_{i_{k-1}=0}^{j-k-I_{k-2}} \left[\frac{\prod_{h=1,h\text{ odd}}^{k-2} \overset{h}{X}_{I_h-I_{k-1}}}{\prod_{g=0,g\text{ even}}^{k-2} \overset{g}{X}_{I_g-I_{k-1}}} \right]^{(-1)^k} \prod_{f=0}^{k-1} \overset{f}{\lambda}{}^{i_f} \tag{2.24a}$$

$$\alpha_0^k = \left(\prod_{h=0}^{k-1} \overset{h}{\lambda} \right)^{-1} \left(\prod_{g=0}^{k-2} \overset{g}{X}_{k+g+1} \right)^{-1}, \tag{2.24b}$$

where $I_k = \sum_{\varsigma=1}^{k} i_\varsigma$, $i_0 = j - k - I_{k-1}$,

$$X_i = \left(\frac{\overline{\overline{x}}}{x} \right)^{\frac{1-(-1)^i}{2}} = \begin{cases} \overline{\overline{x}}/x, & i \text{ odd} \\ 1, & i \text{ even} \end{cases}$$

and we use the notation $\overset{f}{\lambda} = \lambda_{l+f}$.

'As noted after equation (2.22), the δ coefficients pertaining to this hierarchy are obtained from the α coefficients by replacing $\overset{f}{X}_i$ with $1/\overset{f}{X}_i$. Thus, all of the Lax pairs and equations in this hierarchy are completely described by equations (2.14) and (2.24).'

We obtain similar results for the coefficients for the hierarchy corresponding to reductions of the type $x_{m+1,l} = 1/x_{m,l+d}$:

$$\alpha_{j+1}^{k+1} = \sum_{i_1=0}^{j-k} \sum_{i_2=0}^{j-k-I_1} \cdots \sum_{i_{k-1}=0}^{j-k-I_{k-2}} \left[\frac{\prod_{g=0,g\text{ even}}^{k-2} \overset{g}{X}_{I_g-I_{k-1}}}{\prod_{h=1,h\text{ odd}}^{k-2} \overset{h}{X}_{I_h-I_{k-1}}} \right]^{(-1)^k} \prod_{f=0}^{k-1} \overset{f}{\lambda}{}^{i_f} \tag{2.25a}$$

$$\alpha_1^{k+1} = \left(\prod_{h=0}^{k-1} \overset{h}{\lambda} \right)^{-1} \prod_{g=0}^{k-2} \overset{g}{X}_{k+g+1}. \tag{2.25b}$$

In fact, these coefficients α_{j+1}^{k+1} are equal to δ_j^k from the other hierarchy.

These formulae can be used to find any coefficient of interest, which vastly decreases the number of calculations required to find an equation of any order in the hierarchy.

3. A known example

In this section, we will implement formulae (2.24) to explicitly find a known example. The procedure runs as follows:

- First, we decide how many terms we will keep in the Lax pair, i.e. we decide which A_i and D_i will be nonzero, up to $i = m$ say.
- Second, use equation (2.24) to calculate all the coefficients α_j^i up to α_m^{m-1}. It is necessary to calculate every α_j^i with $j \leqslant m$ and $i \leqslant m - 1$ in order to specify the Lax pair.
- Third, calculate the terms A_i and D_i from equations (2.14*a*) and (2.14*b*), noting that any δ_j^i is equal to α_j^i with $X_i, \overline{X}_i, \overline{\overline{X}}_i, \ldots$ replaced with $1/X_i, 1/\overline{X}_i, 1/\overline{\overline{X}}_i, \ldots$. We may then find the corresponding nonlinear equation using the compatibility conditions (2.7) and (2.8).

For our example, we shall retain only those terms A_i, D_i, with $0 \leqslant i \leqslant 3$, which causes there to be two terms in each entry of the N matrix of the Lax pair (see (2.2b)). A Lax pair of this form was already presented in [2] where it was shown to correspond to qP_{II}, we expect the same to occur here.

The next step is to calculate the coefficients α_j^i, δ_j^i up to $i = 2$ and $j = 3$. We begin with the coefficients α_j^1 and δ_j^1, for which inspection of equations (2.11) and (2.12) indicates

$$\alpha_j^1 = \delta_j^1 = \lambda^{j-1}. \tag{3.1}$$

Directly from (2.24) we find

$$\alpha_0^2 = -1/(\lambda\bar{\lambda}X_1) \tag{3.2a}$$

$$\alpha_3^2 = \lambda + \bar{\lambda}/X_1. \tag{3.2b}$$

We have now calculated all the coefficients needed for the present example but we will list the next four as well, for future reference:

$$\alpha_4^2 = \lambda^2 + \lambda\bar{\lambda}/X_1 + \bar{\lambda}^2 \tag{3.3a}$$

$$\alpha_5^2 = \lambda^3 + \lambda^2\bar{\lambda}/X_1 + \lambda\bar{\lambda}^2 + \bar{\lambda}^3/X_1 \tag{3.3b}$$

$$\alpha_6^2 = \lambda^4 + \lambda^3\bar{\lambda}/X_1 + \lambda^2\bar{\lambda}^2 + \lambda\bar{\lambda}^3/X_1 + \bar{\lambda}^4 \tag{3.3c}$$

$$\alpha_7^2 = \lambda^5 + \lambda^4\bar{\lambda}/X_1 + \lambda^3\bar{\lambda}^2 + \lambda^2\bar{\lambda}^3/X_1 + \lambda^4\bar{\lambda} + \bar{\lambda}^5/X_1. \tag{3.3d}$$

At this point we use the coefficients to calculate the values of the nonzero terms in the N matrix. Since A_0 and A_3 are at the ends of the sequence, we can calculate their values directly from (2.7), using the appropriate values of i in that equation. Trivially, these are found to be $A_0 = a_0 =$ constant and $A_3 = \bar{x}b_3 = T_2\sigma\bar{x}/x$ where T_2 is an arbitrary period-two function of l and $\sigma = q^l$. The lower case a_0 and b_3 are the original variables in the N matrix, see (2.2b). Using (2.14a)

$$-A_2 = \alpha_0^2 A_0 + \alpha_3^2 A_3 = -\frac{a_0 x}{\lambda\bar{\lambda}\bar{\bar{x}}} + \left(\lambda + \frac{\bar{\lambda}x}{\bar{\bar{x}}}\right)\frac{T_2\sigma\bar{x}}{x}. \tag{3.4}$$

We then use the coefficients from the previous step to calculate A_1:

$$-A_1 = \alpha_0^1 A_0 + \alpha_2^1 A_2 + \alpha_3^1 A_3 = a_0\left(\frac{1}{\lambda} + \frac{x}{\bar{\lambda}\bar{\bar{x}}}\right) - \frac{\lambda\bar{\lambda}T_2\sigma\bar{x}}{\bar{\bar{x}}}. \tag{3.5}$$

Finally, we can obtain the related equation by substituting these values into (2.7) at $i = 3$, whence we recover $\mathrm{qP_{II}}$ as expected. The form of the equation is

$$\bar{y}\underline{y} = \frac{1 - T_2 r y}{y(\gamma y - \overline{T}_2 r)}, \tag{3.6}$$

where $\log r = \gamma_0 + \gamma_1(-1)^l - ql/2$, $\gamma, \gamma_i =$ constant, and $y = \bar{\bar{x}}/\bar{x}$. Actually, this version of $\mathrm{qP_{II}}$ contains more parameters than those found in [1, 2] as γ and T_2, which are described after the Lax pair below, were not present in those papers. The corresponding Lax pair is

$$L = \begin{pmatrix} \bar{x}/x & -k/(\lambda x) \\ -k\bar{x}/\lambda & 1 \end{pmatrix},$$

$$N = \begin{pmatrix} a_0 + k^2\frac{a_0 x}{\lambda\bar{\lambda}\bar{\bar{x}}} - \left(\lambda + \frac{\bar{\lambda}x}{\bar{\bar{x}}}\right)\frac{T_2\sigma\bar{x}}{x}k^2 & k\left(\frac{x}{\bar{\lambda}\bar{\bar{x}}} - \frac{a_0}{\bar{x}}\left(\frac{1}{\lambda}\right) + \frac{\lambda\bar{\lambda}T_2\sigma}{\bar{\bar{x}}}\right) + k^3\frac{T_2\sigma}{x} \\ k\left(\frac{\bar{\bar{x}}}{\bar{\lambda}x} - d_0\bar{x}\left(\frac{1}{\lambda}\right) + \lambda\bar{\lambda}\overline{T}_2\sigma\bar{\bar{x}}\right) + \overline{T}_2\sigma x k^3 & d_0 + k^2\frac{d_0\bar{\bar{x}}}{\lambda\bar{\lambda}x} - k^2\left(\lambda + \frac{\bar{\lambda}\bar{\bar{x}}}{x}\right)\frac{\overline{T}_2\sigma x}{\bar{x}} \end{pmatrix}.$$

The terms in the N matrix are related to those in (3.6) by $\gamma = d_0/a_0$, T_2 is an arbitrary, period-two function of l and $r = \lambda\bar{\lambda}\bar{\bar{\lambda}}\sigma/a_0$ where $\sigma = q^l$. The spectral parameter is n and it enters the Lax pair via $k = q^n$.

4. Higher order equations

Now that the formulae for finding all the equations in the hierarchy have been derived and their use explained, we will write down some higher order equations and their associated Lax pairs. Section 4.1 will deal with equations obtained from the LMKdV equations via reductions of the type $x_{m+1,l} = x_{m,l+d}$ and section 4.2 will deal with the type $x_{m+1,l} = 1/x_{m,l+d}$.

4.1. Equations corresponding to reductions of the type $x_{m+1,l} = x_{m,l+d}$

This subsection pertains to higher order equations that can be obtained from the LMKdV equation by using a reduction of the type $\hat{x} = x_{l+d}$, where d is some positive integer. Let us first list the equations that will be derived in this section, after which we will write some of the details of their derivation:

$$x_{m+1,l} = x_{m,l+3}: \quad \underline{w}\bar{w} = \frac{1 + T_2 r w}{\overline{\overline{T}}_2 r + \gamma w} \quad (\mathrm{qP_V})$$

$$x_{m+1,l} = x_{m,l+4}: \quad \bar{\bar{\bar{y}}}\bar{y}y\underline{y}\underline{\underline{y}} = \frac{1 - T_2 r \bar{y} y \underline{y}}{\gamma \bar{y} y \underline{y} - \overline{\overline{T}}_2 r}$$

$$x_{m+1,l} = x_{m,l+5}: \quad \bar{\bar{z}}\underline{\underline{z}} = \frac{1}{z}\frac{1 + T_2 r \bar{z}\underline{z}}{\gamma \bar{z}\underline{z} + \overline{\overline{T}}_2 r}.$$

To find these equations we proceed in the same manner as in section 3: first calculate the required coefficients, then use the coefficients to construct the Lax pair and finally use the compatibility condition for the Lax pair to find the resulting equation and conditions on the non-autonomous terms. The coefficients, α_j^3, that will be required for all of the equations presented in this section are calculated using (2.24) and are listed below. Note that these must be used in conjunction with the coefficients found previously (3.2, 3.3).

$$\alpha_0^3 = 1/(\lambda\bar{\lambda}\bar{\bar{\lambda}}\overline{X}_1)$$

$$\alpha_4^3 = \lambda + \bar{\lambda}X_1 + \frac{\bar{\bar{\lambda}}X_1}{\overline{X}_1}$$

$$\alpha_5^3 = \lambda^2 + \lambda\bar{\lambda}X_1 + \frac{\bar{\lambda}\bar{\bar{\lambda}}}{\overline{X}_1} + \frac{\lambda\bar{\bar{\lambda}}X_1}{\overline{X}_1} + \bar{\lambda}^2 + \bar{\bar{\lambda}}^2$$

$$\alpha_6^3 = \lambda^3 + \lambda^2\bar{\lambda}X_1 + \frac{\lambda^2\bar{\bar{\lambda}}X_1}{\overline{X}_1} + \lambda\bar{\lambda}^2 + \frac{\lambda\bar{\lambda}\bar{\bar{\lambda}}}{\overline{X}_1}$$
$$\lambda\bar{\bar{\lambda}}^2 + \bar{\lambda}^3 X_1 + \frac{\bar{\lambda}^2\bar{\bar{\lambda}}X_1}{\overline{X}_1} + \bar{\lambda}\bar{\bar{\lambda}}^2 X_1 + \frac{\bar{\bar{\lambda}}^3 X_1}{\overline{X}_1}.$$

The Lax pair and associated equation that is achieved by truncating the series A_i at A_4 was derived in [2]. Noting that $X_1 = \bar{\bar{x}}/x$, we can use the coefficients α_j^i to find the Lax pair for this $\mathrm{qP_V}$ equation through (2.14a). The Lax pair lies below and the relationships between the terms in the Lax pair and those in equation (4.3) follow

$$L = \begin{pmatrix} \bar{x}/x & -k/(\lambda x) \\ -k\bar{x}/\lambda & 1 \end{pmatrix} \tag{4.1}$$

$$N_{11} = a_0 + k^2 a_0\left(\frac{x}{\lambda\bar{\lambda}\bar{\bar{\bar{x}}}} + \frac{\bar{x}}{\bar{\lambda}\bar{\bar{\lambda}}\bar{\bar{\bar{x}}}} + \frac{x\bar{x}}{\lambda\bar{\bar{\lambda}}\bar{\bar{x}}\bar{\bar{\bar{x}}}}\right) + \overline{T}_2\sigma\left(\frac{\lambda\bar{\lambda}\bar{\bar{x}}}{x} + \frac{\lambda\bar{\bar{\lambda}}\bar{x}\bar{\bar{x}}}{x\bar{\bar{\bar{x}}}} + \frac{\bar{\lambda}\bar{\bar{\lambda}}\bar{x}}{\bar{\bar{\bar{x}}}}\right) + k^4\overline{T}_2\sigma$$

$$N_{12} = -ka_0\left(\frac{1}{\lambda\bar{x}} + \frac{x}{\bar{\lambda}\bar{x}\bar{\bar{x}}} + \frac{x}{\bar{\bar{\lambda}}\bar{\bar{x}}\bar{\bar{\bar{x}}}}\right) - k\overline{T}_2\sigma\frac{\lambda\bar{\lambda}\bar{\bar{\lambda}}}{\bar{\bar{\bar{x}}}} - \frac{k^3 a_0}{\lambda\bar{\lambda}\bar{\bar{\lambda}}\bar{\bar{\bar{x}}}} + \overline{T}_2\sigma\left(\frac{\lambda}{\bar{x}} + \frac{\bar{\lambda}\bar{\bar{x}}}{x\bar{x}} + \frac{\bar{\bar{\lambda}}\bar{\bar{x}}}{x\bar{\bar{\bar{x}}}}\right)$$

$$N_{21} = -kd_0\left(\frac{\bar{x}}{\lambda} + \frac{\bar{x}\bar{\bar{x}}}{\bar{\lambda}x} + \frac{\bar{\bar{x}}\bar{\bar{\bar{x}}}}{\bar{\bar{\lambda}}x}\right) - kT_2\sigma\lambda\bar{\lambda}\bar{\bar{\lambda}}\bar{\bar{\bar{x}}} - \frac{k^3d_0\bar{\bar{\bar{x}}}}{\lambda\bar{\lambda}\bar{\bar{\lambda}}} + T_2\sigma\left(\lambda\bar{x} + \frac{\bar{\lambda}x\bar{x}}{\bar{\bar{x}}} + \frac{\bar{\bar{\lambda}}x\bar{\bar{\bar{x}}}}{\bar{x}}\right)$$

$$N_{22} = d_0 + k^2d_0\left(\frac{\bar{\bar{x}}}{\lambda\bar{\lambda}x} + \frac{\bar{\bar{\bar{x}}}}{\bar{\lambda}\bar{\bar{\lambda}}\bar{x}} + \frac{\bar{\bar{x}}\bar{\bar{\bar{x}}}}{\lambda\bar{\bar{\lambda}}x\bar{x}}\right) + T_2\sigma\left(\frac{\lambda\bar{\lambda}x}{\bar{\bar{x}}} + \frac{\lambda\bar{\bar{\lambda}}x\bar{\bar{\bar{x}}}}{\bar{x}\bar{\bar{x}}} + \frac{\bar{\lambda}\bar{\bar{\lambda}}\bar{\bar{\bar{x}}}}{\bar{x}}\right) + k^4T_2\sigma,$$

where $\lambda = \lambda(l)$, $k = k_0q^n$, with n being the spectral parameter and $\sigma = q^l$. The compatibility condition for this Lax pair produces a series of equations that are either identities or one of two slightly different copies of qP_V. These two copies of qP_V are equal if $q\bar{\bar{\bar{\lambda}}} = \lambda$, there are no other restrictions on the parameters. To get from the form of qP_V that comes directly from the Lax pair to the form as listed in (4.3), we set $r = \lambda\bar{\lambda}\bar{\bar{\lambda}}\bar{\bar{\bar{\lambda}}}\sigma/a_0$, $\gamma = d_0/a_0$ and T_2 remains as it is. The equation is

$$qP_V: \quad \underline{w}\bar{w} = \frac{1 + T_2rw}{\overline{T}_2r + \gamma w}, \tag{4.2}$$

where $w = \bar{\bar{\bar{x}}}/\bar{x}$, γ = constant, T_2 is an arbitrary period-two function and $\log r = \gamma_0 + \gamma_1 j_3^l + \gamma_2 j_3^{2l} - ql/3$, with γ_ι = constant and $j_3^3 = 1$.

This type of condition on λ is common to every equation that has been calculated by the author. Indeed it is expected that, when considering a Lax pair with N matrix truncated at A_j, λ must satisfy $q\lambda(l + j - 1) = \lambda(l)$ and $r = \frac{\sigma}{a_0}\prod_{h=0}^{j}\lambda(l + h)$. We also point out that if one were only interested in this Lax pair and equation, the coefficients α_j^3 with $j > 4$ would be superfluous; they are written above because they are required to find subsequent Lax pairs and equations listed below.

Continuing to the next level in the hierarchy, we obtain the coefficients

$$\alpha_0^4 = -1/(\lambda\bar{\lambda}\bar{\bar{\lambda}}\bar{\bar{\bar{\lambda}}}X_1\overline{X}_1)$$

$$\alpha_5^4 = \lambda + \frac{\bar{\lambda}}{X_1} + \frac{\bar{\bar{\lambda}}\overline{X}_1}{X_1} + \frac{\bar{\bar{\bar{\lambda}}}\overline{X}_1}{X_1\overline{\overline{X}}_1}$$

$$\alpha_6^4 = \lambda^2 + \frac{\lambda\bar{\lambda}}{X_1} + \frac{\lambda\bar{\bar{\lambda}}\overline{X}_1}{X_1} + \frac{\lambda\bar{\bar{\bar{\lambda}}}\overline{X}_1}{X_1\overline{\overline{X}}_1} + \bar{\lambda}^2 + \bar{\lambda}\bar{\bar{\lambda}}\overline{X}_1 + \frac{\bar{\lambda}\bar{\bar{\bar{\lambda}}}\overline{X}_1}{\overline{\overline{X}}_1} + \bar{\bar{\lambda}}^2 + \frac{\bar{\bar{\lambda}}\bar{\bar{\bar{\lambda}}}}{\overline{\overline{X}}_1} + \bar{\bar{\bar{\lambda}}}^2.$$

Ceasing at A_5 yields a new Lax pair for a known fourth-order equation [2]. The L matrix in the Lax pair, as always, is as in (4.2) and the top-left entry of the N matrix is

$$\begin{aligned} N_{11} = a_0 + k^2a_0&\left(\frac{x}{\lambda\bar{\lambda}\bar{\bar{x}}} + \frac{x\bar{x}}{\lambda\bar{\bar{\lambda}}\bar{\bar{x}}\bar{\bar{\bar{x}}}} + \frac{x\bar{\bar{x}}}{\lambda\bar{\bar{\bar{\lambda}}}\bar{\bar{\bar{x}}}\overset{4}{x}} + \frac{\bar{x}}{\bar{\lambda}\bar{\bar{\lambda}}\bar{\bar{\bar{x}}}} + \frac{\bar{x}\bar{\bar{x}}}{\bar{\lambda}\bar{\bar{\bar{\lambda}}}\bar{\bar{\bar{x}}}\overset{4}{x}} + \frac{\bar{\bar{x}}}{\bar{\bar{\lambda}}\bar{\bar{\bar{\lambda}}}\overset{4}{x}}\right) \\ &- k^2\overline{T}_2\sigma\left(\lambda\bar{\lambda}\bar{\bar{\lambda}}\frac{\bar{\bar{\bar{x}}}}{x} + \lambda\bar{\lambda}\bar{\bar{\bar{\lambda}}}\frac{\bar{\bar{x}}\bar{\bar{\bar{x}}}}{x\overset{4}{x}} + \lambda\bar{\bar{\lambda}}\bar{\bar{\bar{\lambda}}}\frac{\bar{x}\bar{\bar{x}}}{x\overset{4}{x}} + \bar{\lambda}\bar{\bar{\lambda}}\bar{\bar{\bar{\lambda}}}\frac{\bar{x}}{\overset{4}{x}}\right)\frac{k^4a_0x}{\lambda\bar{\lambda}\bar{\bar{\lambda}}\bar{\bar{\bar{\lambda}}}\overset{4}{x}} \\ &- k^4\overline{T}_2\sigma\left(\frac{\lambda\bar{x}}{x} + \frac{\bar{\lambda}\bar{x}}{\bar{\bar{x}}} + \frac{\bar{\bar{\lambda}}\bar{\bar{\bar{x}}}}{\bar{\bar{x}}} + \frac{\bar{\bar{\bar{\lambda}}}\bar{\bar{\bar{x}}}}{\overset{4}{x}}\right). \end{aligned}$$

We omit the rest of the N matrix for brevity, however it is easily calculated from the information already given.

The associated equation, given below, is a reduction of the LMKdV equation under $\hat{x} = \overset{4}{x}$. The notation $\overset{4}{x} = x_{l+4}$ is used instead of $\bar{\bar{\bar{\bar{x}}}}$ because too many bars become difficult to read

$$\bar{\bar{y}}\bar{y}y\underline{y}\underline{\underline{y}} = \frac{1 - T_2r\bar{y}y\underline{y}}{\gamma\bar{y}y\underline{y} - \overline{T}_2r}, \tag{4.3}$$

where $y = \bar{\bar{\bar{x}}}/\bar{\bar{x}}$, $\log r = \gamma_0 + \gamma_1 i^l + \gamma_2(-1)^l + \gamma_3(-i)^l - ql/4$ and $\gamma_\iota =$ constant. The relationship between r and quantities in the Lax pair for this equation is $r = \lambda\bar{\lambda}\bar{\bar{\lambda}}\bar{\bar{\bar{\lambda}}}\overset{4}{\lambda}\sigma/a_0$ and $q\overset{4}{\lambda} = \lambda$ to ensure compatibility.

We will list one more higher order equation. The next set of coefficients that we require is

$$\alpha_0^5 = 1\big/\big(\lambda\bar{\lambda}\bar{\bar{\lambda}}\bar{\bar{\bar{\lambda}}}\overset{4}{\lambda}\overline{X}_1\overline{\overline{X}}_1\big)$$

$$\alpha_6^5 = \lambda + \bar{\lambda}X_1 + \frac{\bar{\bar{\lambda}}X_1}{\overline{X}_1} + \frac{\bar{\bar{\bar{\lambda}}}X_1\overline{\overline{X}}_1}{\overline{X}_1} + \frac{\overset{4}{\lambda}X_1\overline{\overline{X}}_1}{\overline{X}_1\overline{\overline{\overline{X}}}_1},$$

where $X_1 = \bar{\bar{x}}/x$ as usual. These coefficients lead to a Lax pair with the L matrix as before (see (4.2)) and the top-left entry of the N matrix below, again the other parts of N are omitted:

$$\begin{aligned}
N_{11} = a_0 + k^2 a_0 \Bigg(& \frac{x\bar{x}}{\lambda\overset{4}{\lambda}\overset{4}{x}\overset{5}{x}} + \frac{x}{\lambda\bar{\lambda}\bar{\bar{x}}} + \frac{\bar{\bar{x}}}{\bar{\bar{\lambda}}\bar{\bar{\bar{\lambda}}}\overset{4}{x}} + \frac{\bar{x}\bar{\bar{x}}}{\bar{\lambda}\overset{4}{\lambda}\overset{4}{x}\overset{5}{x}} + \frac{\bar{x}}{\bar{\lambda}\bar{\bar{\lambda}}\bar{\bar{\bar{x}}}} + \frac{x\bar{x}}{\lambda\bar{\bar{\lambda}}\bar{\bar{\bar{x}}}\overset{4}{x}} + \frac{\bar{\bar{\bar{x}}}}{\bar{\bar{\bar{\lambda}}}\overset{4}{\lambda}\overset{5}{x}} \\
& + \frac{x\bar{x}}{\lambda\bar{\bar{\lambda}}\bar{\bar{x}}\bar{\bar{\bar{x}}}} + \frac{\bar{x}\bar{\bar{x}}}{\bar{\lambda}\bar{\bar{\lambda}}\bar{\bar{\bar{x}}}\overset{4}{x}} + \frac{\bar{\bar{\bar{x}}}}{\bar{\bar{\bar{\lambda}}}\overset{4}{\lambda}\overset{5}{x}} \Bigg) + k^2\overline{T}_2\sigma\Bigg(\lambda\bar{\lambda}\bar{\bar{\lambda}}\bar{\bar{\bar{\lambda}}}\frac{\overset{4}{x}}{x} + \bar{\lambda}\bar{\bar{\lambda}}\bar{\bar{\bar{\lambda}}}\overset{4}{\lambda}\frac{\bar{x}}{\overset{5}{x}} + \lambda\bar{\bar{\lambda}}\bar{\bar{\bar{\lambda}}}\overset{4}{\lambda}\frac{\bar{x}\bar{\bar{x}}}{x\overset{5}{x}} \\
& + \lambda\bar{\lambda}\bar{\bar{\lambda}}\overset{4}{\lambda}\frac{\bar{\bar{\bar{x}}}\overset{4}{x}}{x\overset{5}{x}} + \lambda\bar{\lambda}\bar{\bar{\bar{\lambda}}}\overset{4}{\lambda}\frac{\bar{\bar{x}}\bar{\bar{\bar{x}}}}{x\overset{5}{x}}\Bigg) + k^4 a_0\Bigg(\frac{x}{\lambda\bar{\lambda}\bar{\bar{\lambda}}\bar{\bar{\bar{\lambda}}}\overset{4}{x}} + \frac{\bar{x}}{\bar{\lambda}\bar{\bar{\lambda}}\bar{\bar{\bar{\lambda}}}\overset{4}{\lambda}\overset{5}{x}} + \frac{x\bar{x}}{\lambda\bar{\bar{\lambda}}\bar{\bar{\bar{\lambda}}}\overset{4}{\lambda}\bar{\bar{x}}\overset{5}{x}} \\
& + \frac{x\bar{\bar{\bar{x}}}}{\lambda\bar{\lambda}\bar{\bar{\lambda}}\overset{4}{\lambda}\overset{4}{x}\overset{5}{x}} + \frac{x\bar{\bar{\bar{x}}}}{\lambda\bar{\lambda}\bar{\bar{\bar{\lambda}}}\overset{4}{\lambda}\bar{\bar{x}}\overset{5}{x}}\Bigg) + k^4\overline{T}_2\sigma\Bigg(\frac{\lambda\overset{4}{\lambda}\bar{x}\overset{4}{x}}{x\overset{5}{x}} + \frac{\bar{\bar{\lambda}}\bar{\bar{\bar{\lambda}}}\overset{4}{x}}{\bar{\bar{x}}} + \frac{\bar{\lambda}\overset{4}{\lambda}\bar{x}\overset{4}{x}}{\bar{\bar{x}}\overset{5}{x}} + \frac{\lambda\bar{\lambda}\bar{\bar{x}}}{x} \\
& + \frac{\lambda\bar{\bar{\lambda}}\bar{x}}{\bar{\bar{\bar{x}}}} + \frac{\lambda\bar{\bar{\bar{\lambda}}}\bar{x}\overset{4}{x}}{x\overset{4}{x}} + \frac{\bar{\lambda}\bar{\bar{\bar{\lambda}}}\bar{x}\overset{4}{x}}{\bar{\bar{x}}\bar{\bar{\bar{x}}}} + \frac{\lambda\bar{\bar{\lambda}}\bar{x}\bar{\bar{x}}}{x\bar{\bar{\bar{x}}}} + \frac{\bar{\bar{\bar{\lambda}}}\overset{4}{\lambda}\bar{\bar{\bar{x}}}}{\overset{5}{x}} + \frac{\bar{\bar{\lambda}}\overset{4}{\lambda}\bar{\bar{\bar{x}}}\overset{4}{x}}{\bar{\bar{x}}\overset{5}{x}}\Bigg) + k^6\overline{T}_2\sigma.
\end{aligned}$$

Incredibly this cumbersome Lax pair has as its compatibility condition the following, rather simple, fourth-order equation:

$$\bar{\bar{z}}\underline{\underline{z}} = \frac{1}{z}\frac{1 + T_2 r\bar{z}\underline{z}}{\gamma\bar{z}\underline{z} + \overline{T}_2 r}, \tag{4.4}$$

where $z = \overset{4}{x}/\bar{\bar{x}}$ and $\log r = -ql/5 + \gamma_0 + \gamma_1 j_5^l + \gamma_2 j_5^{2l} + \gamma_3 j_5^{3l} + \gamma_4 j_5^{4l}$, with $\gamma_i =$ constant, $j_5^5 = 1$, and T_2 is an arbitrary, period-two function of l.

4.2. *Equations corresponding to reductions of the type $x_{m+1,l} = 1/x_{m,l+d}$*

Here, we will write down some equations, with their Lax pairs, from the hierarchy that arises from the LMKdV equation via reductions of the type $x_{m+1,l} = 1/x_{m,l+d}$, $d =$ constant:

$$x_{m+1,l} = 1/x_{m,l+1}: \quad x\bar{\bar{x}} = \frac{1 + \overline{T}_2 r\bar{x}^2}{\gamma\bar{x}^2 + T_2 r} \qquad (\mathrm{qP_{III}})$$

$$x_{m+1,l} = 1/x_{m,l+2}: \quad \bar{y}\underline{y} = y\frac{1 - \overline{T}_2 r y}{\gamma y - T_2 r} \qquad (\mathrm{qP_{II}})$$

$$x_{m+1,l} = 1/x_{m,l+3}: \quad x\overset{4}{x} = \frac{1 + \overline{T}_2 r\bar{x}\bar{\bar{x}}}{\gamma\bar{x}\bar{\bar{x}} + T_2 r}.$$

The procedure used to obtain these results is just the same as that explained in section 3. However, as outlined in section 2.2, now $A_1 = \beta_1 x\bar{x}$, $\beta_1 =$ constant, is an end point of the series of terms in the Lax pairs, and the odd and even powers of k have been redistributed. We will not list the coefficients used in finding the results presented here because they are easily obtained from those used in section 4.1. To find α_j^k as required with the present hierarchy, use α_{j-1}^{k-1} from section 4.1 and replace $\overset{h}{X}_i \to 1/\overset{h}{X}_i$. That being the case, we directly move to the first Lax pair which has L as in (4.2) and

$$N = \begin{pmatrix} k^2\left(\frac{\beta_1 x\bar{x}}{\lambda} + \lambda\overline{T}_2\sigma\frac{\bar{x}}{x}\right) & k\beta_1 x + k^3\lambda\overline{T}_2\frac{\sigma}{x} \\ k\frac{\beta_2}{x} + T_2\sigma x & k^2\left(\frac{\beta_2}{\lambda x\bar{x}} + \lambda T_2\sigma\frac{x}{\bar{x}}\right) \end{pmatrix}$$

and $\lambda = q\bar{\lambda}$ for compatibility so $r = \lambda\bar{\lambda}\sigma/\beta_2 = \gamma_0 q^{-l}$ with $\gamma_0 =$ constant. This leads to the first non-trivial equation in this part of the hierarchy, $\mathrm{qP_{III}}$:

$$x\bar{\bar{x}} = \frac{1 + \overline{T}_2 r\bar{x}^2}{\gamma\bar{x}^2 + T_2 r}. \tag{4.5}$$

This equation and Lax pair was found in [2], except here the equation has two extra free parameters coming from the T_2 term which is an arbitrary period-two function of l and $\gamma =$ constant.

The N matrix of the next Lax pair is

$$N_{11} = k^2\left[\beta_1\bar{x}\left(\frac{x}{\lambda} + \frac{\bar{\bar{x}}}{\bar{\lambda}}\right) - \lambda\bar{\lambda}\overline{T}_2\sigma\frac{\bar{\bar{x}}}{x}\right] + k^4\overline{T}_2\sigma$$

$$N_{12} = k\beta_1 x + k^3\left[\beta_1\frac{\bar{x}\bar{\bar{x}}}{\lambda\bar{\lambda}} - \left(\lambda + \bar{\lambda}\frac{\bar{\bar{x}}}{x}\right)\overline{T}_2\sigma\right]$$

$$N_{21} = k\beta_2 x + k^3\left[\frac{\beta_2}{\lambda\bar{\lambda}\bar{x}\bar{\bar{x}}} - \left(\lambda + \bar{\lambda}\frac{x}{\bar{\bar{x}}}\right)\overline{T}_2\sigma\right]$$

$$N_{22} = k^2\left[\frac{\beta_1}{\bar{x}}\left(\frac{1}{\lambda x} + \frac{1}{\bar{\lambda}\bar{\bar{x}}}\right) - \lambda\bar{\lambda}\overline{T}_2\sigma\frac{x}{\bar{\bar{x}}}\right] + k^4\overline{T}_2\sigma,$$

where $\beta_1, \beta_2 =$ constant and, to ensure compatibility, $\lambda = q\bar{\bar{\lambda}}$. We set $r = \lambda\bar{\lambda}\bar{\bar{\lambda}}\sigma/\beta_2$ causing $\log r = \gamma_0 + \gamma_1(-1)^l - ql/2$, since $\sigma = q^l$. The resulting equation is an alternative $\mathrm{qP_{II}}$. After setting $y = \bar{\bar{x}}\bar{x}$,

$$\bar{y}\underline{y} = y\frac{1 - \overline{T}_2 r y}{\gamma y - T_2 r}. \tag{4.6}$$

The final Lax pair that will be presented from this part of the hierarchy has the same L matrix again (4.2) and the components of the N matrix are

$$N_{11} = -k^2\left[\beta_1\left(\frac{x\bar{x}}{\lambda} + \frac{\bar{x}\bar{\bar{x}}}{\bar{\lambda}} + \frac{\bar{\bar{x}}\bar{\bar{\bar{x}}}}{\bar{\bar{\lambda}}}\right) + \lambda\bar{\lambda}\bar{\bar{\lambda}}\overline{T}_2\sigma\frac{\bar{\bar{\bar{x}}}}{x}\right] - k^4\left[\beta_1\frac{x\bar{\bar{\bar{x}}}}{\lambda\bar{\lambda}\bar{\bar{\lambda}}} + \left(\lambda\frac{\bar{x}}{x} + \bar{\lambda}\frac{\bar{x}}{\bar{\bar{x}}} + \bar{\bar{\lambda}}\frac{\bar{\bar{\bar{x}}}}{\bar{\bar{x}}}\right)\overline{T}_2\sigma\right]$$

$$N_{12} = k\beta_1 x + k^3\left[\beta_1\left(\frac{\bar{\bar{x}}}{\lambda\bar{\lambda}} + \frac{x\bar{\bar{\bar{x}}}}{\bar{\lambda}\bar{\bar{\lambda}}\bar{x}} + \frac{\bar{\bar{x}}\bar{\bar{\bar{x}}}}{\lambda\bar{\bar{\lambda}}\bar{x}}\right) + \left(\frac{\lambda\bar{\lambda}}{\bar{\bar{x}}} + \frac{\lambda\bar{\bar{\lambda}}\bar{\bar{\bar{x}}}}{\bar{x}\bar{\bar{x}}} + \frac{\bar{\lambda}\bar{\bar{\lambda}}\bar{\bar{\bar{x}}}}{x\bar{x}}\right)\overline{T}_2\sigma\right] + k^5\overline{T}_2\frac{\sigma}{x}$$

$$N_{21} = k\frac{\beta_2}{x} + k^5 T_2\sigma x + k^3\left[\beta_2\left(\frac{1}{\lambda\bar{\lambda}\bar{\bar{x}}} + \frac{\bar{x}}{\bar{\lambda}\bar{\bar{\lambda}}x\bar{\bar{\bar{x}}}} + \frac{\bar{x}}{\lambda\bar{\bar{\lambda}}\bar{\bar{x}}\bar{\bar{\bar{x}}}}\right) + \left(\lambda\bar{\lambda}\bar{\bar{x}} + \frac{\lambda\bar{\bar{\lambda}}\bar{x}\bar{\bar{x}}}{\bar{\bar{\bar{x}}}} + \frac{\bar{\lambda}\bar{\bar{\lambda}}x\bar{x}}{\bar{\bar{\bar{x}}}}\right)T_2\sigma\right]$$

$$N_{22} = -k^2\left[\beta_2\left(\frac{1}{\lambda x\bar{x}} + \frac{1}{\bar{\lambda}\bar{x}\bar{\bar{x}}} + \frac{1}{\bar{\bar{\lambda}}\bar{\bar{x}}\bar{\bar{\bar{x}}}}\right) + \lambda\bar{\lambda}\bar{\bar{\lambda}}T_2\sigma\frac{x}{\bar{\bar{\bar{x}}}}\right] - k^4\left[\frac{\beta_2}{\lambda\bar{\lambda}\bar{\bar{\lambda}}x\bar{\bar{\bar{x}}}} + \left(\frac{\lambda x}{\bar{x}} + \frac{\bar{\lambda}\bar{\bar{x}}}{\bar{x}} + \frac{\bar{\bar{\lambda}}\bar{\bar{x}}}{\bar{\bar{\bar{x}}}}\right)T_2\sigma\right].$$

With this member of the hierarchy we require $\lambda = q\overline{\overline{\overline{\lambda}}}$ for compatibility, which causes $\log r = \gamma_0 + \gamma_1 j_3^l + \gamma_2 j_3^{2l} - ql/3$, $\gamma_\iota =$ constant and $j_3^3 = 1$. Compatibility of L and N shows that this is a Lax pair for the fourth-order equation

$$x\overset{4}{x} = \frac{1 + \overline{T}_2 r\bar{x}\bar{\bar{x}}}{\gamma\bar{x}\bar{\bar{\bar{x}}} + T_2 r}, \tag{4.7}$$

where $\gamma = \beta_1/\beta_2 =$ constant and $r = \lambda\bar{\lambda}\bar{\bar{\lambda}}\bar{\bar{\bar{\lambda}}}\sigma/\beta_2$.

5. Conclusion

In this paper, we have presented two new hierarchies of nonlinear q-difference equations, one of which includes $\mathrm{qP_{II}}$ and $\mathrm{qP_{V}}$, the other of which includes $\mathrm{qP_{III}}$ in addition to higher order equations. The relationship between the equations in each hierarchy was found using a series of Lax pairs and, as such, a Lax pair accompanies each equation in the hierarchy. All of the resulting equations are non-autonomous and contain multiple free parameters while each Lax pair is 2×2.

Even though these Lax pairs increase in complexity at each level of the hierarchy, the equations retain the same simple structure while increasing the order and number of free parameters. The persistence of a simple structure in the equations may facilitate the discovery of special solutions applicable to all members of the hierarchy.

We must point out that some key features of the method used to establish the hierarchy have not been proven in generality. We simply conjecture their validity based on agreement with results.

We note that these hierarchies have their roots in reductions from the lattice modified KdV equation, it remains to be seen whether similar results lie behind other partial difference equations. It would eventually be interesting to find reductions from lattice equations to the q-Garnier hierarchy constructed by Sakai in [24].

At this point there is still a significant deficiency in knowledge about the generic solutions of q-Painlevé equations. The author is unaware of any instances where Birkhoff's theory of linear q-difference equations has been applied to deduce information about the solutions of q-Painlevé equations. The question of the global properties of solutions remains completely open.

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